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# Iterative Solutions of Boundary Integral Equations of the First Kind

JOHN F. AHNER

*Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240*

AND

GEORGE C. HSIAO

*Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716*

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In this paper the Landweber–Fridman iterative scheme is used to construct the solution to a system of Fredholm integral equations of the first kind. In general, the underlying integral operator is not in a form for which the Landweber–Fridman iterative scheme can be used directly. Consequently, appropriate modifications are given. The method presented is then applied to the exterior Dirichlet problem for Laplace's equation in $\mathbb{R}^2$. Examples illustrating the method are given. © 1992 Academic Press, Inc.

## 1. INTRODUCTION

The purpose of this paper is to apply the Landweber–Fridman iterative scheme [15, 5] to a system of Fredholm integral equations of the first kind. In particular, we construct the solution of the system

$$-\frac{1}{2\pi}\int_\Gamma \sigma(y)\ln|x-y|\,ds_y-\omega=f(x), \qquad x\in\Gamma,$$

$$\int_\Gamma \sigma(y)\,ds_y=b,$$

where $\Gamma$ is the simple, closed, smooth boundary of a bounded domain $\Omega$ in $\mathbb{R}^2$. Here $f$ and $b$ are given, and $\sigma$ and $\omega$ are unknowns, all satisfying some regularity conditions to be specified later. Boundary integral equations of this kind arise in many applications, for example, in conformal



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mapping, potential flow, and plane elasticity to name a few (see, e.g., Hsiao and MacCamy [10]).

It was shown by Hsiao and Wendland [11] that the boundary integral operator

$$V\sigma(x) := -\frac{1}{2\pi}\int_\Gamma \sigma(y) \ln |x-y| \, ds_y, \qquad x \in \Gamma,$$

is positive definite (see (3.2)) and hence nonnegative definite, provided the diameter of $\Omega$ is less than unity. However, $V$ is not nonnegative definite in general. Hence the Landweber–Fridman iterative scheme cannot be applied to the aforementioned directly without some modifications.

We organize the paper as follows. First, under the assumption that the diameter of $\Omega$ is less than unity, we reformulate the system of integral equations as a uniquely solvable Fredholm integral equation of the first kind. Then, using the Landweber–Fridman iterative scheme, we show that this integral equation can be solved iteratively.

Next we consider the case when the diameter is greater than or equal to unity. We show that this case can be reduced to the previous situation. Thus we obtain an analytical representation for the solution without assuming a priori that the diameter is less than unity. We apply our results to obtain the solution to the exterior Dirichlet problem for Laplace's equation in $\mathbb{R}^2$.

In the next section we introduce our system of boundary integral equations. In Section 3 we give certain properties of our underlying integral operator. In Section 4 we reformulate our system of integral equations as a uniquely solvable Fredholm integral equation of the first kind. We demonstrate that this integral equation can be solved iteratively without imposing any restrictions on the diameter of our underlying domain. In Section 5 we apply our results to the exterior Dirichlet potential problem in $\mathbb{R}^2$. In Section 6 we extend the results in Section 4 to a modification of the integral equation considered in Section 4. In the last two sections we illustrate our method for certain exterior Dirichlet potential problems. In Section 7 we consider the case when the underlying boundary is a circle. In Section 8 we consider the case when the underlying boundary is an ellipse.

## 2. A System of Boundary Integral Equations

We begin with some notation and definitions. Let $\Omega$ be a bounded, simply connected domain in $\mathbb{R}^2$ containing the origin with a smooth boundary $\Gamma$. Let $\Omega^c$ denote the region exterior to $\bar{\Omega}$. Let $\hat{n}$ denote the unit normal to $\Gamma$ directed into $\Omega^c$. Let $x$ and $y$ denote typical points in $\mathbb{R}^2$.

We introduce the following standard integral operators of potential theory:

$$V\sigma(x) := -\frac{1}{2\pi}\int_\Gamma \sigma(y)\ln|x-y|\,ds_y, \qquad x\in\mathbb{R}^2, \tag{2.1}$$

$$K\sigma(x) := -\frac{1}{2\pi}\int_\Gamma \sigma(y)\frac{\partial}{\partial n(y)}\ln|x-y|\,ds_y, \qquad x\in\Gamma, \tag{2.2}$$

$$K'\sigma(x) := -\frac{1}{2\pi}\int_\Gamma \sigma(y)\frac{\partial}{\partial n(x)}\ln|x-y|\,ds_y, \qquad x\in\Gamma. \tag{2.3}$$

For $s\geqslant 0$, we denote by $H^s(\Gamma)$, the Sobolev spaces of real valued, generalized functions on $\Gamma$, and for $s<0$, let $H^s(\Gamma)$ denote the dual of $H^{-s}(\Gamma)$. The norm in $H^s(\Gamma)$ will be denoted by $\|\cdot\|_s$. There is one exception to our notation. In the case $s=0$ we denote $H^s(\Gamma)$ by $L^2(\Gamma)$. Thus

$$(\sigma,\mu)_0=\int_\Gamma \sigma(y)\,\mu(y)\,ds_y, \tag{2.4}$$

$$\|\sigma\|_0=(\sigma,\sigma)_0^{1/2}, \tag{2.5}$$

are the corresponding $L^2$ inner product and $L^2$ norm, respectively. Let $\mathring{H}^s(\Gamma) := \{\sigma\in H^s(\Gamma)\,|\,(1,\sigma)_0=0\}$.

It can be shown (see, e.g., [12]) that $V$, $K$, $K'$: $H^s(\Gamma)\to H^{s+1}(\Gamma)$ continuously, and moreover, they are compact on $H^s(\Gamma)$. It is well known that $V$ is self-adjoint and that $K$ and $K'$ are adjoint with respect to the inner product $(\cdot,\cdot)_0$.

Let

$$d := \text{diameter}(\Omega). \tag{2.6}$$

For the case when $0<d<1$, it is known (see, e.g., [11, 8]) that $V$ is an isomorphism from $H^s(\Gamma)$ onto $H^{s+1}(\Gamma)$ for all $s\in\mathbb{R}$; moreover, it satisfies the strong coercivity condition

$$(V\sigma,\sigma)_0 \geqslant \gamma\,\|\sigma\|^2_{-1/2}, \tag{2.7}$$

for some constant $\gamma>0$. Thus when $0<d<1$, the integral equation

$$V\sigma=f \tag{2.8}$$

has a unique solution $\sigma\in H^s(\Gamma)$ corresponding to a given function $f\in H^{s+1}(\Gamma)$.

For general boundaries, however, the unique solvability of the integral equation (2.8) is not assured. With respect to the terminology of Jaswon and Symm [14], the boundary of $\Omega$ is a "$\Gamma$-contour" if there exists a nontrivial solution to $V\sigma = 0$. With respect to the function space $H^{-1/2}(\Gamma)$, a necessary and sufficient condition for this nonuniqueness to occur is when the transfinite diameter associated with $\Gamma$, which we denote by $C_\Gamma$, is equal to one. It is known (see, e.g., [6, Chap. 16]) that the transfinite diameter of $\Gamma$ can never exceed its Euclidean diameter. Consequently, the boundary of $\Omega$ is not a $\Gamma$-contour when $0 < d < 1$.

We now state the following important result that can be found in [10, 11].

THEOREM 2.1. *There exists a unique solution pair $(\sigma, \omega) \in H^s(\Gamma) \times \mathbb{R}$ to the system of equations*

$$V\sigma(x) - \omega = f(x), \qquad x \in \Gamma, \tag{2.9a}$$

$$(\sigma, 1)_0 = b, \tag{2.9b}$$

*for any given $(f, b) \in H^{s+1}(\Gamma) \times \mathbb{R}$ for any given $s \in \mathbb{R}$.*

In particular we denote by $(e_\Gamma, \omega_\Gamma)$ the unique solution of (2.9) for $(f, b) = (0, 1)$. $\omega_\Gamma$ is usually referred to as the Robin constant (see, e.g., [18, 9, 19]), and $e_\Gamma(x)$ is referred to as either the natural layer (see [21, 9]) or the equilibrium distribution (see [19]). It can easily be shown that the function $e_\Gamma$ also satisfies the integral equation of the second kind

$$(\tfrac{1}{2}I + K')\, e_\Gamma(x) = 0, \qquad x \in \Gamma, \tag{2.10}$$

which has a unique solution subject to the constraint $(e_\Gamma, 1)_0 = 1$. Since the function $e_\Gamma$ plays an important part in the subsequent analysis, we point out for the interested reader that $e_\Gamma$ can be constructed by an iterative scheme applied to the integral equation (2.10). (See [1] for the corresponding case in $\mathbb{R}^3$.)

In Section 4 we shall construct the solution to the system of boundary integral equations (2.9) for the case when $f \in H^1(\Gamma)$.

## 3. SOME PROPERTIES OF $V$

We now collect some basic results for the operator $V$ defined by (2.1).

THEOREM 3.1. *Let $\Gamma$ be a smooth boundary.*

(A) *The operator $V$ is a compact, self-adjoint operator on $H^s(\Gamma)$ for all $s \in \mathbb{R}$.*

(B) *$V$: $H^s(\Gamma) \to H^{s+1}(\Gamma)$ continuously for all $s \in \mathbb{R}$.*

(C) *The operator $V$ is $\mathring{H}^{-1/2}(\Gamma)$-elliptic, that is,*

$$(V\sigma, \sigma)_0 \geqslant \gamma \|\sigma\|^2_{-1/2}, \qquad \textit{for all} \quad \sigma \in \mathring{H}^{-1/2}(\Gamma), \tag{3.1}$$

*for some constant $\gamma > 0$ independent of $\sigma$, and hence $V$ is positive definite on $\mathring{H}^{-1/2}(\Gamma)$ in the sense that*

$$(V\sigma, \sigma)_0 \geqslant 0 \qquad \textit{and} \qquad (V\sigma, \sigma)_0 = 0 \textit{ iff } \sigma = 0 \tag{3.2}$$

*for all $\sigma \in \mathring{H}^{-1/2}(\Gamma)$.*

(D) *If $0 < d < 1$, then*

$$(V\sigma, \sigma)_0 \geqslant \gamma \|\sigma\|^2_{-1/2}, \qquad \textit{for all} \quad \sigma \in H^{-1/2}(\Gamma), \tag{3.3}$$

*for some constant $\gamma > 0$ independent of $\sigma$. Thus $V$ is positive definite on $H^{-1/2}(\Gamma)$ and hence on $L^2(\Gamma)$ in the sense of* (3.2).

The results in this theorem are well known (see, e.g., [11, 12]). We note that we have already given some of these results in the previous section, however, we have included them again in the above theorem for future convenience. Throughout the remainder of this paper, unless otherwise stated, we will take $L^2(\Gamma)$ to be the underlying Hilbert space for $V$.

In the next theorem, we establish some spectral properties of $V$ which motivate certain results we obtain in the examples considered in Sections 7 and 8. The theorem also shows that the result in Theorem 3.1(D) is not true for general boundaries and that without some geometric restrictions placed on $\Gamma$, $V$ is not positive definite on $L^2(\Gamma)$. It is precisely this observation that motivates some our ideas in the next section.

THEOREM 3.2. (A) *If $0 < C_\Gamma < 1$, then $V$ is positive definite.*

(B) *If $C_\Gamma = 1$, then $\omega_\Gamma = 0$ and* $\dim N(V) = 1$, *where $N$ denotes the nullspace.*

(C) *If $1 < C_\Gamma < \infty$, then $V$ has exactly one negative eigenvalue $\lambda$ and* $\dim N(\lambda I - V) = 1$.

*Proof.* (A) The validity of this part is known (see, e.g., [11, 19]).

(B) In this case the boundary is a $\Gamma$-contour and the proof of this part is well known (see, e.g., [14]).

(C) First let us show that there exists a negative eigenvalue. It is known (see, e.g., [9, 19]) that $2\pi\omega_\Gamma = -\ln C_\Gamma$. Thus $(Ve_\Gamma, e_\Gamma)_0 < 0$. Since $V$ is a compact, self-adjoint operator on $L^2(\Gamma)$, it follows that there exists at least one negative eigenvalue $\lambda_1$.

Suppose $\lambda_1$ and $\lambda_2$ are two different negative eigenvalues of $V$ and let $\phi_1$ and $\phi_2$ be corresponding eigenfunctions. Thus $(\phi_1, \phi_2)_0 = 0$ and from

Theorem 3.1(C), $(\phi_i, 1)_0 \neq 0$, $i = 1, 2$. Define the function $\psi$ by $\psi := c_1\phi_1 + c_2\phi_2$ where $c_1$, $c_2$ are nonzero real constants and chosen such that $(\psi, 1)_0 = 0$. From Theorem 3.1(C) it follows that $(V\psi, \psi)_0 > 0$. From the orthogonality of $\phi_1$ and $\phi_2$, however, it follows that

$$(V\psi, \psi)_0 = c_1^2\lambda_1(\phi_1, \phi_1)_0 + c_2^2\lambda_2(\phi_2, \phi_2)_0 < 0,$$

and we have a contradiction.

Finally, to show that $\dim N(\lambda I - V) = 1$, we assume that there are two linearly independent eigenfunctions $\phi_1$ and $\phi_2$. From the Gram–Schmidt process we may assume that these eigenfunctions are orthogonal. Then using essentially the same argument we just gave, it follows that $\dim N(\lambda I - V) = 1$. ∎

Let $\mu(V)$ denote the spectrum of $V$ and let $\mu_C(V)$, $\mu_P(V)$, and $\mu_R(V)$ denote the continuous spectrum, point spectrum, and residual spectrum of $V$, respectively. Since $V$ is a bounded, self-adjoint operator on $L^2(\Gamma)$, $\mu_R(V)$ is empty. Since $V$ is a compact operator on $L^2(\Gamma)$, $\lambda = 0 \in \mu(V)$ and $\mu(V)\setminus\{0\}$ consists of at most a countable set of eigenvalues with $\lambda = 0$ as the only possible limit point. Denote the eigenvalues of $V$ by $\{\lambda_n\}_{n=0}^{\infty}$ and the corresponding set of orthogonal eigenfunctions by $\{\psi_n(x)\}_{n=0}^{\infty}$.

We finally give a result which we will need which is a consequence of Theorem 3.1 and the spectral radius formula for bounded, self-adjoint operators.

LEMMA 3.3. *Let $0 < d < 1$. The eigenvalues of $V$ may be ordered*

$$\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots > 0,$$

*where*

$$\|V\|_0 = \lambda_0.$$

## 4. ITERATIVE SOLUTIONS

In this section we obtain an analytical representation of the solution to the system of integral equations in (2.9) by using the Landweber–Fridman iterative scheme [15, 5]. Our representation is based on the following result:

LEMMA 4.1. *Let $A$ be a compact, linear, self-adjoint, positive definite operator on a real Hilbert space $H$. Consider the operator equation*

$$Au = g, \tag{4.1}$$

*where $g \in H$ is a given function. Let $v$ be a real number such that $0 < v < 2/\|A\|$. Suppose $g$ is such that (4.1) has a solution $u$. Then $u$ is unique and the sequence obtained from the iterative scheme*

$$\begin{aligned} u_{n+1} &= u_n + v(g - Au_n), \qquad n = 0, 1, 2, ..., \\ u_0 &= vg, \end{aligned} \tag{4.2}$$

*converges strongly to $u$, i.e.,* $\lim_{n \to \infty} \|u_n - u\|_H = 0$.

We remark that the uniqueness of $u$ follows from the assumed positive definiteness of $A$. A proof for the strong convergence of $\{u_n\}$ to $u$ can be found in [4].

Next consider the system of integral equations of Section 2,

$$V\sigma(x) - \omega = f(x), \qquad x \in \Gamma, \tag{4.3a}$$

$$(\sigma, 1)_0 = b, \tag{4.3b}$$

where $(f, b)$ is given and $(\sigma, \omega)$ is to be found. We now take $f \in H^1(\Gamma)$. From Theorem 2.1 it follows that there exists a unique solution $(\sigma, \omega) \in L^2(\Gamma) \times \mathbb{R}$.

We now reformulate (4.3) into a form for which the Landweber–Fridman iterative scheme may be used to construct the solution. To this end multiply (4.3a) by the natural layer $e_\Gamma$ and then integrate over the boundary. We find after some simplification

$$\omega = \omega_\Gamma b - (f, e_\Gamma)_0. \tag{4.4}$$

It follows that the solution $\sigma$ must satisfy

$$V\sigma(x) = g(x), \qquad x \in \Gamma, \tag{4.5a}$$

$$(\sigma, 1)_0 = b, \tag{4.5b}$$

where $g(x)$ is a known function defined by

$$g(x) := f(x) + \omega_\Gamma b - (f, e_\Gamma)_0. \tag{4.6}$$

Two observations follow immediately from the unique solvability of (4.3). First, the system (4.5) has a unique solution. Second, $g \in V(L^2(\Gamma))$; that is, the function $g$ defined in (4.6) lies in the range of $V$ taken with respect to $L^2(\Gamma)$.

The following result can now be established from Theorem 3.1 and the unique solvability of (4.5):

THEOREM 4.2. *Let $0 < d < 1$. Consider*

$$V\sigma(x) = g(x), \qquad x \in \Gamma,$$

*where $g(x)$ is defined in* (4.6). *Then there exists a unique solution $\sigma \in L^2(\Gamma)$ to the integral equation. Moreover, $\sigma$ satisfies the compatibility condition* $(\sigma, 1)_0 = b$.

The following result can now be easily established from Theorem 3.1, Lemma 3.3, Lemma 4.1, and Theorem 4.2:

THEOREM 4.3. *Let $0 < d < 1$. Let $f \in H^1(\Gamma)$ and let $g$ be defined as in* (4.6). *Let $\nu$ be a real number such that $0 < \nu < 2/\lambda_0$. Then the integral equation* (4.5a) *has a unique solution $\sigma \in L^2(\Gamma)$. Let*

$$\sigma_n(x) = \nu \sum_{k=0}^{n} (I - \nu V)^k g(x), \qquad x \in \Gamma. \tag{4.7}$$

*Then the sequence $\{\sigma_n(x)\}$ converges strongly to the solution $\sigma(x)$ in $L^2(\Gamma)$.*

Next consider the case $d \geqslant 1$. We will show that this case can be reduced to the previous analysis by using an idea in [8] (see also [19]). Let $D$ be a positive real constant such that $D > d$. Transform $\Gamma$ to a contour $\Gamma'$ by

$$\Gamma' = \Gamma / D. \tag{4.8}$$

Let $\Omega'$ denote the domain enclosed by $\Gamma'$ and let $d' :=$ diameter$(\Omega')$. Then $0 < d' < 1$. Let $x = Dx'$, $y = Dy'$.

Then the system of integral equations (4.5) becomes

$$-\frac{D}{2\pi} \int_{\Gamma'} \ln |x' - y'| \, \sigma(Dy') \, ds_{y'} - \frac{b}{2\pi} \ln D = g(Dx'), \qquad x' \in \Gamma', \tag{4.9a}$$

$$D \int_{\Gamma'} \sigma(Dy') \, ds_{y'} = b. \tag{4.9b}$$

Consequently, the system in (4.9) can be written

$$V_{\Gamma'} \tilde{\sigma}(x') = \tilde{g}(x'), \qquad x' \in \Gamma', \tag{4.10a}$$

$$\int_{\Gamma'} \tilde{\sigma}(y') \, ds_{y'} = b, \tag{4.10b}$$

where $V_{\Gamma'}$ denotes the single layer potential with boundary $\Gamma'$, $\tilde{\sigma}(x') := D\sigma(Dx')$, and $\tilde{g}(x') := g(Dx') + (b \ln D)/2\pi$ is a known function.

Since $\sigma \in L^2(\Gamma)$ is the unique solution of (4.5) it follows that $\tilde{\sigma} \in L^2(\Gamma')$ is the unique solution to (4.10). Furthermore, since $0 < d' < 1$, theorems analogous to Theorems 4.2 and 4.3 hold for the integral equation (4.10a). To avoid a tedious repetition we will not repeat the appropriate modifications of Theorems 4.2 and 4.3 for the system (4.10). We note, however, that the parameter $\nu$ appearing in Theorem 4.3 must now satisfy $0 < \nu < 2/\|V_{\Gamma'}\|$.

## 5. The Exterior Dirichlet Potential Problem

With regard to the system of boundary integral equations in (4.3), we consider as a model problem the following exterior Dirichlet problem (P) for Laplace's equation in $\mathbb{R}^2$:

$$\Delta u(x) = 0, \qquad x \in \Omega^c, \tag{5.1a}$$

$$u(x) = h(x), \qquad x \in \Gamma, \tag{5.1b}$$

$$u(x) = O(1) \qquad \text{as} \quad |x| \to \infty, \tag{5.1c}$$

where $\Delta$ denotes the Laplacian operator and $h(x)$ is a given function. It is well known (see, e.g., [20]) that (P) has a unique weak solution $u \in H^1_{\text{loc}}(\Omega^c)$ for given $h \in H^{1/2}(\Gamma)$.

From the generalized Green's formula (see, e.g., [3]), we have the following representation for the solution of (P):

$$\omega - \frac{1}{2\pi} \int_\Gamma \left[ u(y) \frac{\partial}{\partial n(y)} \ln |x - y| - \frac{\partial u(y)}{\partial n} \ln |x - y| \right] ds_y$$

$$= \begin{cases} u(x) & \text{in } \Omega^c, \\ \frac{1}{2} u(x) & \text{on } \Gamma, \\ 0 & \text{in } \Omega, \end{cases} \tag{5.2}$$

where $\omega$ is a constant which can be determined from $h$ as we will show. In order to use the results in the previous section, we require that $h \in H^1(\Gamma)$.

From the representation formula we obtain the following boundary integral equation for $\sigma := \partial u/\partial n$:

$$V\sigma(x) - \omega = f(x), \qquad x \in \Gamma, \tag{5.3}$$

where

$$f(x) := (-\tfrac{1}{2} I + K)\, h(x). \tag{5.4}$$

From the growth condition of $u$ at infinity in (5.1c), it follows that

$$(\sigma, 1)_0 = 0. \tag{5.5}$$

The system of boundary integral equations (5.3) and (5.5) is precisely of the form in (4.3). Proceeding as in Section 4 leading to (4.4) we find

$$\omega = -(f, e_\Gamma)_0 = (h, e_\Gamma)_0. \tag{5.6}$$

Thus the system (5.3) and (5.5) becomes

$$V\sigma(x) = g(x), \qquad x \in \Gamma, \tag{5.7a}$$

$$(\sigma, 1)_0 = 0, \tag{5.7b}$$

where $g(x)$ is a known function defined by

$$g(x) := (-\tfrac{1}{2}I + K)\, h(x) + (h, e_\Gamma)_0. \tag{5.8}$$

The system (5.7) is of the form in (4.5) and the analysis of the previous section can now be used. If $0 < d < 1$, we have from Theorem 4.3 that the unique solution to (5.7a) can be found by the Landweber–Fridman iterative scheme. If $d \geqslant 1$, we must first rescale and then reformulate (5.7) as an equivalent system of the form (4.10). Finally, the Landweber–Fridman iterative scheme can then be used to construct $\tilde{\sigma}$, from which $\sigma$ is obtained.

For the interested reader we point out that the system (5.3) and (5.5) and hence the system (5.7) also arises in the investigation of the exterior Dirichlet problem for the Helmholtz equation for low frequencies [7]. A similar system arises for the exterior Dirichlet dynamic elasticity problem for low frequencies [13]. In both of these cases the role of $\sigma$ in (5.7) is the zeroth order term of a suitable expansion in terms of the wave number $k$ of $\partial u/\partial n$ in the case of the exterior Helmholtz equation problem and the traction in the case of the exterior elasticity problem. For details, we refer the reader to the papers [7, 13].

In Sections 7 and 8 specific examples for (P) are considered where $\Gamma$ is a circle and an ellipse, respectively. In each example Landweber–Fridman iteration is used to construct the solution to (5.7). Since the iterative scheme can be used both when $0 < d < 1$ and when $d \geqslant 1$, in order to minimize the analytical details in the examples, we assume hereafter that $0 < d < 1$. To facilitate the analysis of the example in Section 8, we introduce in the next section a modified single layer potential.

## 6. A Modified Single Layer Potential

Here we introduce a modified single layer potential integral operator and extend some of the results in Section 4 for it. Let $\rho(x)$ be a smooth function defined on $\Gamma$ which satisfies

$$0 < m \leqslant \rho(x) \leqslant M, \qquad x \in \Gamma, \tag{6.1}$$

where $m$ and $M$ are positive real constants. Define the following modified single layer potential integral operator:

$$\tilde{V}\sigma(x) := -\frac{1}{\rho(x)}\frac{1}{2\pi}\int_\Gamma \ln|x-y|\,\sigma(y)\,ds_y, \qquad x\in\Gamma. \tag{6.2}$$

Let $L^2(\Gamma;\rho)$ denote the Hilbert space of real valued, square integrable functions defined on $\Gamma$ with weight function $\rho$, with $L^2$ inner product, and $L^2$ norm

$$(\sigma,\mu)_\rho = \int_\Gamma \sigma(y)\,\mu(y)\,\rho(y)\,ds_y, \tag{6.3}$$

$$\|\sigma\|_\rho = (\sigma,\sigma)_\rho^{1/2}. \tag{6.4}$$

There is one exception to our notation here. When $\rho(x)\equiv 1$, $L^2(\Gamma;1)=L^2(\Gamma)$ and we denote the corresponding inner product and norm by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ in order to retain their definitions in (2.4) and (2.5), respectively. From (6.1) we have

$$m(\sigma,\sigma)_0 \leqslant (\sigma,\sigma)_\rho \leqslant M(\sigma,\sigma)_0 \tag{6.5}$$

and it follows that

LEMMA 6.1. $\sigma\in L^2(\Gamma)\Leftrightarrow\sigma\in L^2(\Gamma;\rho)$.

Dividing the integral equation (4.5a) by $\rho(x)$ we have

$$\tilde{V}\sigma(x) = g_\rho(x), \qquad x\in\Gamma, \tag{6.6}$$

where

$$g_\rho(x) := g(x)/\rho(x) = \{f(x)+\omega_\Gamma b-(f,e_\Gamma)_0\}/\rho(x) \tag{6.7}$$

from (4.6).

Noting that

$$(\tilde{V}\sigma,\sigma)_\rho = (V\sigma,\sigma)_0, \tag{6.8}$$

the following theorem follows from Theorem 3.1 and Lemma 6.1:

THEOREM 6.2. *Let $0<d<1$. Then $\tilde{V}$ is a compact, self-adjoint, positive definite, linear operator on $L^2(\Gamma;\rho)$.*

From Theorem 4.2 and Lemma 6.1 we have:

THEOREM 6.3. *Let $0<d<1$. Let $f\in H^1(\Gamma)$. Then the integral equation (6.6) has a unique solution $\sigma\in L^2(\Gamma;\rho)$.*

Denote the eigenvalues of $\tilde{V}$ by $\{\mu_n\}_{n=0}^{\infty}$ and the corresponding eigenfunctions by $\{\phi_n(x)\}_{n=0}^{\infty}$. Corresponding to Lemma 3.3 we have

LEMMA 6.4. *Let $0<d<1$. The eigenvalues of $\tilde{V}$ may be ordered*

$$\mu_0 \geqslant \mu_1 \geqslant \mu_2 \geqslant \cdots > 0,$$

*where*

$$\|\tilde{V}\|_\rho = \mu_0.$$

Corresponding to Theorem 4.3, the next result follows from Lemma 4.1, Lemma 6.1, Theorem 6.2, Theorem 6.3, and Lemma 6.4:

THEOREM 6.5. *Let $0<d<1$. Let $f \in H^1(\Gamma)$. Let $\nu$ be a real number such that $0<\nu<2/\mu_0$. Then the integral equation (6.6) has a unique solution $\sigma \in L^2(\Gamma;\rho)$. Let*

$$\tilde{\sigma}_n(x) := \nu \sum_{k=0}^{\infty} (I-\nu\tilde{V})^k g_\rho(x), \qquad x \in \Gamma. \tag{6.9}$$

*Then the sequence $\{\tilde{\sigma}_n(x)\}$ converges strongly to the solution $\sigma(x)$ in $L^2(\Gamma;\rho)$.*

## 7. AN EXAMPLE WHEN $\Gamma$ IS A CIRCLE

In this section the results of Section 4 are used to construct the solution to an exterior Dirichlet problem for Laplace's equation for the case when $\Gamma$ is a circle of radius $a$. Let $(r_x, \theta_x)$ denote the polar coordinates of the point $x$. For a circle of radius $a$, $ds = a\,d\theta$. From [16, p. 259]

$$-\ln|x-y| = -\ln r_> + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{r_<}{r_>}\right)^m \cos m(\theta_x - \theta_y), \tag{7.1}$$

where $r_> = \max\{r_x, r_y\}$, $r_< = \min\{r_x, r_y\}$, and $r_< < r_>$.

Let us now determine some spectral properties for $V$, for the case when $\Gamma$ is a circle of radius $a$, which we will need in the subsequent analysis. We begin by computing the eigenvalues and corresponding eigenfunctions of $V$. Letting $\psi_0(x) \equiv 1$, $\psi_n^c(x) = \cos n\theta_x$, $\psi_n^s(x) = \sin n\theta_x$, it follows from (7.1) and the orthogonality of the trigonometric functions that

$$V\psi_0(x) = -(a \ln a)\,\psi_0(x), \qquad x \in \Gamma, \tag{7.2}$$

$$V\psi_n^p(x) = \frac{a}{2n}\,\psi_n^p(x), \qquad p = c, s,\ n \in \mathbb{Z}^+,\ x \in \Gamma. \tag{7.3}$$

Thus each function in the set $\{1, \cos n\theta_x, \sin n\theta_x, n \in \mathbb{Z}^+\}$ is an eigenfunction of $V$. Since this set is a complete orthogonal set in $L^2[0, 2\pi]$, it follows that this set contains the only eigenfunctions of $V$.

From the above results and because $V$ is a compact, self-adjoint operator on $L^2(\Gamma)$, we have

LEMMA 7.1. (A) *If* $0 < a < 1$, *then* $V$ *has only positive eigenvalues and* $0 \in \mu_C(V)$.

(B) *If* $a = 1$, *then* $V$ *has only nonnegative eigenvalues and* $0 \in \mu_P(V)$. *Moreover*, $\dim N(V) = 1$.

(C) *If* $1 < a$, *then* $V$ *has exactly one negative eigenvalue* $\lambda_-$ *and all the other eigenvalues are strictly positive. Furthermore*, $\dim N(\lambda_- I - V) = 1$ *and* $0 \in \mu_C(V)$.

It is known that the transfinite diameter of a circle of radius $a$ is $a$. Thus it is seen that the above result agrees with our general result in Theorem 3.2.

We now impose the condition that $0 < d < 1$. Thus

$$0 < a < 1/2. \tag{7.4}$$

For these values of $a$, from (7.2) and (7.3) it can be shown

$$\|V\|_0 = \max_{n \in \mathbb{Z}^+} \left\{ -a \ln a, \frac{a}{2n} \right\} = -a \ln a. \tag{7.5}$$

Consider now the following exterior Dirichlet potential problem ($\mathrm{P}_C$) for a circle of radius $a = 1/4$:

$$\begin{aligned} \Delta u(x) &= 0, \qquad r_x > 1/4, \\ u(x) &= \tfrac{1}{4} \cos \theta_x, \qquad r_x = 1/4, \\ u(x) &= O(1) \qquad \text{as} \quad r_x \to \infty. \end{aligned}$$

From our results in Section 5, $\sigma := \partial u/\partial n$ satisfies the integral equation (5.7a) and in terms of the notation of that section, $h(x) = (\cos \theta_x)/4$. Let us now compute the function $g(x)$ defined in (5.8). From (7.2) and from the definition of $e_\Gamma$ we find

$$e_\Gamma(x) \equiv \frac{2}{\pi}. \tag{7.6}$$

Thus $(h, e_\Gamma)_0 = 0$. For $x, y \in \Gamma$, it can be shown

$$-\frac{\partial}{\partial n(y)} \ln |x - y| = -\frac{1}{2a}\bigg|_{a=1/4} = -2, \tag{7.7}$$

from which it follows $Kh(x) \equiv 0$ on $\Gamma$. Thus

$$g(x) = (-\tfrac{1}{2}I + K)\,h(x) + (h, e_\Gamma)_0 = -\tfrac{1}{8}\cos\theta_x. \tag{7.8}$$

To implement Theorem 4.3 to construct $\sigma(x)$ we must choose $\nu$ such that

$$0 < \nu < 2/\|V\|_0 \approx 5.77, \tag{7.9}$$

where we have used (7.5) in the last step. For convenience we take $\nu = 1$.
From (7.3) and (7.8) we have

$$(I - V)\,g(x) = (\tfrac{7}{8})(-\tfrac{1}{8}\cos\theta_x), \tag{7.10}$$

and from an induction argument we have

$$(I - V)^k\,g(x) = (\tfrac{7}{8})^k\,(-\tfrac{1}{8}\cos\theta_x), \qquad k = 0, 1, 2, .... \tag{7.11}$$

Consequently, from Theorem 4.3 and (7.11) we obtain

$$\sigma(x) = -\cos\theta_x, \qquad x \in \Gamma. \tag{7.12}$$

Now let us find the solution to ($P_C$) in $\Omega^c$. For $x \in \Omega^c$ and $y \in \Gamma$, we have from (7.1)

$$-\ln|x - y| = -\ln r_x + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{1}{4r_x}\right)^m \cos m(\theta_x - \theta_y), \tag{7.13}$$

$$-\frac{\partial}{\partial n(y)}\ln|x - y| = \frac{1}{r_x}\sum_{m=1}^{\infty}\left(\frac{1}{4r_x}\right)^{m-1} \cos m(\theta_x - \theta_y). \tag{7.14}$$

From (5.2), (5.6), (7.12), (7.13), (7.14), and the orthogonality of the trigonometric functions, it can be shown that

$$u(x) = \frac{\cos\theta_x}{16r_x}, \qquad x \in \Omega^c. \tag{7.15}$$

## 8. An Example when $\Gamma$ is an Ellipse

In this section the results of Section 6 are used to construct the solution to an exterior Dirichlet problem for Laplace's equation for the case when $\Gamma$ is an ellipse. The elliptical coordinates $(\mu_x, \phi_x)$ of the point $x$ are related to the rectangular coordinates $(x_1, x_2)$ by the transformation

$$x_1 = \frac{c}{2}\cosh\mu_x\cos\phi_x, \qquad x_2 = \frac{c}{2}\sinh\mu_x\sin\phi_x, \tag{8.1}$$

where $0 \leqslant \mu_x < \infty$, $0 \leqslant \phi_x \leqslant 2\pi$. The closed curves corresponding to $\mu = \text{constant} > 0$, $0 \leqslant \phi \leqslant 2\pi$ are confocal ellipses of interfocal distance $c$, eccentricity $e = (\cosh \mu)^{-1}$, major axis $c \cosh \mu$, and minor axis $c \sinh \mu$. The limiting case $\mu = 0$ represents the line segment between the foci.

In this section $\Gamma$ will denote the ellipse corresponding to $\mu = b$, $0 \leqslant \phi \leqslant 2\pi$, where $b$ is some constant. To avoid the degenerate case, we will assume that $b > 0$. Letting

$$\rho(y) := \frac{c}{2} [\sinh^2 b + \sin^2 \phi_y]^{1/2}, \tag{8.2}$$

it can be shown

$$ds_y = \rho(y)\, d\phi_y, \tag{8.3}$$

and it is seen that $\rho$ satisfies an inequality of the form (6.1). From [17, p. 1202] we have

$$\begin{aligned} -\ln |x-y| = -[\mu_> + \ln(c/4)] + \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\mu_>} \\ \times [\cosh m\mu_< \cos m\phi_x \cos m\phi_y + \sinh m\mu_< \sin m\phi_x \sin m\phi_y], \end{aligned} \tag{8.4}$$

where $\mu_> = \max\{\mu_x, \mu_y\}$, $\mu_< = \min\{\mu_x, \mu_y\}$, and $\mu_< < \mu_>$.

With $\rho$ defined in (8.2), let $\tilde{V}$ denote the modified single layer potential as defined in (6.2). Let us now compute the eigenvalues and corresponding eigenfunctions of $\tilde{V}$. Letting $\phi_0(x) := 1/\rho(x)$, $\phi_n^c(x) := (\cos n\phi_x)/\rho(x)$, $\phi_n^s(x) := (\sin n\phi_x)/\rho(x)$, it can be shown,

$$\tilde{V}\phi_0(x) = -[b + \ln(c/4)]\, \phi_0(x), \tag{8.5}$$

$$\tilde{V}\phi_n^c(x) = \frac{1}{n} (e^{-nb} \cosh nb)\, \phi_n^c(x), \qquad n \in \mathbb{Z}^+, \tag{8.6}$$

$$\tilde{V}\phi_n^s(x) = \frac{1}{n} (e^{-nb} \sinh nb)\, \phi_n^s(x), \qquad n \in \mathbb{Z}^+, \tag{8.7}$$

where $x \in \Gamma$. From the completeness of the set of functions $\{1, \cos n\phi, \sin n\phi, n \in \mathbb{Z}^+\}$ in $L^2[0, 2\pi]$, it follows that $\{\phi_0(x), \phi_n^c(x), \phi_n^s(x), n \in \mathbb{Z}^+\}$ are the only eigenfunctions of $\tilde{V}$.

From Section 6 we have that $\tilde{V}$ is a compact, self-adjoint operator on $L^2(\Gamma; \rho)$. From the above results it then follows that

LEMMA 8.1. *Let $b$ and $c$ be positive constants.*

(A) *If* $-[b+\ln(c/4)]>0$, *then* $\tilde{V}$ *has only positive eigenvalues and* $0\in\mu_C(\tilde{V})$.

(B) *If* $-[b+\ln(c/4)]=0$, *then* $\tilde{V}$ *has only nonnegative eigenvalues and* $0\in\mu_P(\tilde{V})$. *Moreover*, $\dim N(\tilde{V})=1$.

(C) *If* $-[b+\ln(c/4)]<0$, *then* $\tilde{V}$ *has exactly one negative eigenvalue* $\lambda_-$, *and all the remaining eigenvalues are strictly positive. Furthermore*, $\dim N(\lambda_- I-\tilde{V})=1$ *and* $0\in\mu_C(\tilde{V})$.

A similar result was derived in the previous section for $V$ for the case when $\Gamma$ was a circle. In what follows we now impose the condition $0<d<1$. Thus $b$ and $c$ must satisfy

$$0<c\cosh b<1. \tag{8.8}$$

From (8.5)–(8.8), it can be shown

$$\|\tilde{V}\|_\rho=\max_{n\in\mathbb{Z}^+}\left\{-[b+\ln(c/4)],\frac{1\pm e^{-2nb}}{2n}\right\}=-[b+\ln(c/4)]. \tag{8.9}$$

Consider now the following exterior Dirichlet problem ($\mathrm{P}_E$) for an ellipse corresponding to $b=1$ and $c=1/2$, which clearly satisfy inequality (8.8):

$$\begin{aligned}\Delta u(x)&=0, && x\in\Omega^c,\\ u(x)&=\cos\phi_x, && x\in\Gamma,\\ u(x)&=O(1), && \text{as } |x|\to\infty.\end{aligned}$$

From our results in Section 6, $\sigma:=\partial u/\partial n$ satisfies the integral equation (6.6). With $h(x)=\cos\phi_x$, let us now determine the function $g_\rho(x)$ defined in (6.7). From (8.5) and the definition of $e_\Gamma(x)$ we find

$$e_\Gamma(x)=\frac{1}{2\pi}\frac{1}{\rho(x)}. \tag{8.10}$$

Thus $(h,e_\Gamma)_0=0$. From [2] we have

$$K(\cos\phi_x)=-\tfrac{1}{2}e^{-2}\cos\phi_x. \tag{8.11}$$

Thus

$$\begin{aligned}g_\rho(x)&=[(-\tfrac{1}{2}I+K)\,h(x)+(h,e_\Gamma)_0]/\rho(x)\\ &=-[(1+e^{-2})\cos\phi_x]/2\rho(x).\end{aligned} \tag{8.12}$$

To implement Theorem 6.5 to construct $\sigma(x)$ we must choose $\nu$ such that

$$0<\nu<2/\|\tilde{V}\|_\rho\approx 1.85, \tag{8.13}$$

where we have used (8.9) in this last step. For convenience we take $\nu=1$.

From (8.6) and (8.12) we obtain

$$(I-\tilde{V})\, g_\rho(x) = \left(\frac{1-e^{-2}}{2}\right) g_\rho(x), \tag{8.14}$$

and from an induction argument we have

$$(I-\tilde{V})^k\, g_\rho(x) = \left(\frac{1-e^{-2}}{2}\right)^k g_\rho(x), \qquad k=0, 1, 2, .... \tag{8.15}$$

Consequently, from Theorem 6.5, (8.12), and (8.15) we obtain

$$\sigma(x) = -(\cos\phi_x)/\rho(x). \tag{8.16}$$

Now let us find the solution to $(\mathrm{P}_E)$ in $\Omega^c$. For $x \in \Omega^c$ and $y \in \Gamma$, it follows from (8.4)

$$\begin{aligned} -\ln|x-y| = -(\mu_x - \ln 8) + \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\mu_x} \\ \times [\cosh m\mu_y \cos m\phi_x \cos m\phi_y + \sinh m\mu_y \sin m\phi_x \sin m\phi_y], \end{aligned} \tag{8.17}$$

where $\mu_y = b = 1$. Furthermore, from [2], it follows that

$$\begin{aligned} -\frac{\partial}{\partial n(y)} \ln|x-y| = \frac{2}{\rho(y)} \sum_{m=1}^{\infty} e^{-m\mu_x} [\sinh m\mu_y \cos m\phi_x \cos m\phi_y \\ + \cosh m\mu_y \sin m\phi_x \sin m\phi_y]. \end{aligned} \tag{8.18}$$

From (5.2), (5.6), (8.16), (8.17), (8.18), and the orthogonality of the trigonometric functions, we find

$$u(x) = e^{b-\mu_x} \cos\phi_x, \qquad x \in \Omega^c. \tag{8.19}$$

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